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Bayesian Methods in Psychological Research: The case of IRT.

Métodos Bayesianos en la investigación psicológica: el caso de la teoría de respuesta al ítem.

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ABSTRACT

Bayesian methods have become increasingly popular in social sciences due to its flexibility in accommodating numerous models from different fields. The domain of item response theory is a good example of fruitful research, incorporating in the lasts years new developments and models, which are being estimated using the Bayesian approach. This is partly because of the availability of free software such as WinBUGS and R, which has permitted researchers to explore new possibilities.

In this paper we outline the Bayesian inference for some IRT models. It is briefly explained how the Bayesian method works. The implementation of Bayesian estimation in conventional software is discussed and sets of codes for running the analyses are provided. All the applications are exemplified using simulated and real data sets.

Key words: Item response theory models, Bayesian Inference, WinBUGS.

RESUMEN

Los métodos Bayesianos se han vuelto populares en las ciencias sociales debido a su flexibilidad para acomodar varios modelos provenientes de distintas áreas. El área de la teoría de respuesta al ítem (TRI) ejemplifica como operan los métodos Bayesianos. En particular, la existencia de paquetes estadísticos gratuitos como WinBUGS y R le permite al investigador explorar varias posibilidades.

En este artículo se presentan las inferencias Bayesianas para algunos modelos de TRI y se explica concisamente cómo funcionan los métodos Bayesianos. Se discute la implementación de estos métodos en paquetes estadísticos convencionales y se presentan algunas sintaxis computacionales para hacer análisis Bayesianos. Todas las aplicaciones aquí expuestas se ejemplifican usando datos simulados y reales.

Palabras clave: Teoría de respuesta al ítem, inferencia Bayesiana, WinBUGS.

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The use of psychological tests to measure the level that a person has of certain unobservable trait is a common practice in different areas of the social sciences. These traits or properties can be interpreted, for instance, as intelligence, ability for some subject in the school, and certain physical skill. It is possible to find examples of observable human behavior indicating that a person has more or less of such a general trait. Psychological measurement aims to quantify the amount or magnitude of individual's characteristics by using mathematical (statistical) models. In this context, the family of Item Response Theory models (IRT; Fisher and Molenaar, 1995; De Boeck and Wilson, 2004) is a good example of statistical models that have been successfully utilized in various domains of social sciences. These models relate the probability of observing a certain result (answer to an item, behavior or reaction to a stimulus, number of times a physical activity is performed, etc) to individual parameters and stimulus parameters. On the basis of observations from a population of individuals reacting to a set of stimulus, these parameters are estimated and subsequent inferences are carried out. From a classical inferential point of view, the interest is put in finding the parameter values (estimates) that maximize the likelihood of the data that has been observed. The standard likelihood-based approach seeks for the parameters that are more likely to have generated the observed data, yielding maximum likelihood estimates of the parameters of interest.

The Bayesian approach is an alternative to the standard statistical inference techniques. Its main feature is the capacity to incorporate prior knowledge in the statistical analysis by using prior distributions on the parameters of interest, in a way that will be explained later in the paper. Inferences are then based on samples from the posterior distributions, which can be used to summarize all the necessary information about the parameters of interest. Advances in computational aspects, especially in the implementation of numerical algorithms that permits for sampling from the posterior distributions, and the availability of software to implement them, have increased the popularity of Bayesian inferences in the social sciences community. Recently, Lee and Wagenmakers (2005) argued that Bayesian methods have an important role to play in many psychological problems where standard techniques are difficult or impossible to apply.

The aim of this paper is to serve as a tutorial for the Bayesian estimation of some IRT models. Although IRT models were primary conceived for educational measurement, in which the trait to be measured is the ability an individual has to answer items from a test; its use is nowadays more widespread. In Stockman (1977), for example, the person's parameters are really resolutions regarding decolonization, the test items are delegations at the General Assembly of the United Nations, and the item answers are votes in favor or against those resolutions. In the field of emotions, the constructs measured are anger, irritation, or any other feeling and the stimulus are items in which persons are asked about what would be his/her reaction in different situations. In this paper, examples in both the fields of education and emotions will be given.

The remaining of the paper is organized as follows: first the IRT models are introduced and the standard likelihood-based methods of estimation are briefly discussed. Next, the main ideas behind the Bayesian method are explained, as well as the main advantages of its use. Subsequently, the implementation of Bayesian estimation of IRT models in conventional software is discussed and sets of codes for running the analyses are provided. The paper finalizes with some conclusions and a discussion.

IRT MODELS: MODEL SPECIFICATION AND PARAMETER ESTIMATION

The basic idea behind IRT models is to connect the probability of observing an individual's response to a certain stimulus, to both stimulus characteristics and person's characteristics. In the case of educational measurement, which will be the example used in this paper to introduce the models, the ability one individual has to answer a test item is considered the persons' characteristics. The set of items in a test are considered the stimulus that individuals are confronted to, and the difficulty of theme can be considered as an item characteristic. As it was mentioned, feelings, behaviors or other constructs can be considered in psychological research, in which case the stimulus can be different situations or questionnaires with items asking for feelings one would have under different situations. De Boeck and Wilson (2004) use data on verbal aggression to illustrate several IRT models. This data set is publicly available at http://bear.soe.berkeley.edu/EIRM/ and will be used in this paper to exemplify the Bayesian estimation of IRT models.

In order to make clear the presentation of the models, let us introduce some notation. Suppose that a total of n individuals are confronted to a test composed of k items. Let Y_{ij} be the random variable denoting the answer of person i to item j. When items are dichotomously scored (correct, incorrect) the observed data can take the values $y_{ij} = 1$, for a correct answer, and $y_{ij} = 0$ otherwise. For example $y_{41} = 1$ indicates that the fourth person answered correctly item 1. The probability of a correct answer is denoted by p_{ij}. As pointed out earlier, IRT models link the probability of a correct answer to persons' and items' parameters, which will be denoted by θ_i and β_j, respectively, so that $p_{ij} = \Pr(Y_{ij} = 1 \mid \theta_i) = f(\theta_i, \beta_j)$ for some function $f(.)$. Thus, conditional on the ability, Y_{ij} follows a Bernoulli distribution with parameter p_{ij}. The function $f(.)$ is typically chosen to be the logistic or the standard normal distribution function.We consider here logistic models, in which case $f(t) = \exp(t)/1 + \exp(t)$.

The Rasch model

When the difficulty of the item is the assumed item's characteristic, the resulting IRT model is the Rasch model (Rasch, 1960). The Rasch model is also called the one-parameter logistic (1PL) model because it uses only one parameter per item, the difficulty. The probability of a correct answer is modeled as

$$p_{ij} = \Pr(Y_{ij} = 1 \mid \theta_i) = \frac{\exp(\theta_i - \beta_j)}{1 + \exp(\theta_i - \beta_j)} \qquad (0.1)$$

Other models that consider more than one item parameter are the two-parameter logistic model (2PL), which incorporates an item discriminating parameter to differentiate better between different levels of ability, and the three-parameter logistic model (3PL), which, besides the difficulty and discriminating parameters, includes a guessing parameter which accounts for the possibility to answer the item by guessing. These models will not be considered in this paper, but the codes presented to run the analysis can be easily accommodated to fit them.

The detailed explanation and uses of the above mentioned models are beyond the scope of this paper. For an introduction of IRT models oriented to the psychologists, the reader is refereed to Embretson and Reise (2000).

The Linear Logistic Test Model (LLTM)

Consider a Rasch model in which the difficulty parameters β_j are a linear combination of certain item properties. For instance, in a mathematics test, in order to solve an item like $2 + 3 \times 5$ one needs to master the following three sub-tasks: i) the product is performed before the sum, ii) knowledge of how to multiply $3 \times 5 = 15$, iii) knowledge of how to sum $2 + 15 = 17$. The difficulty of the item is then decomposed as a linear combination of the three item properties. This type of model is known as the Linear Logistic Test Model (LLTM; Fisher, 1973). The probability of a correct answer is modeled as

$$p_{ij} = \Pr(Y_{ij} = 1 \mid \theta_i) = \frac{\exp(\theta_i - \beta_j^*)}{1 + \exp(\theta_i - \beta_j^*)} \qquad (0.2)$$

where $\beta_j^* = \sum_{k=0}^{K} \beta_k X_{jk}$. Following the previous example, when three item properties are considered $\beta_j^* = \beta_0 + \beta_1 X_{j1} + \beta_2 X_{j2} + \beta_3 X_{j3}$, where $X_{jk} = 1$ if sub-task k is needed to solve item j and $X_{jk} = 0$ otherwise. The value of X_{j0} is typically chosen to be 1, acting as an intercept term.

Likelihood function

Both the Rasch and the LLTM models link the probability of a correct answer, p_{ij}, to persons' and items' parameters. Assuming that for each person, their answers to all items in the test are independent given his/her ability, and that the people's responses are independent from each other, the likelihood function reads as

$$L(\theta, \beta \mid y) = \prod_{i=1}^{n} \prod_{j=1}^{k} p_{ij}^{y_{ij}} (1 - p_{ij})^{1 - y_{ij}}. \qquad (0.3)$$

Likelihood-based estimation methods are then used to maximize the likelihood function in order to obtain maximum likelihood estimates of the parameters of interest. The idea is simple: look for the values of θ and β that are more likely to have generated the observed data.

In the likelihood function (1.3), two kind of parameters are involved; the item parameters, which are assumed to be fixed-effects parameters, and the persons parameters. Depending on whether the person parameters are considered as fixed-effects or random-effects parameters, different likelihood-based estimation methods can be considered. Well known likelihood estimation methods in IRT models are Joint Maximum Likelihood (JML), Conditional Maximum Likelihood (CML) and Marginal Maximum Likelihood (MML), being the latter the most popular and used one. The JML method considers both the ability and difficulty parameters as fixed effects, so that the likelihood in Equation (1.3) is jointly maximized with respect to item and person parameters. In the CML approach, after conditioning on the sum raw score (i.e., the total number of correct responses one individual obtains), the person's parameters disappear in the conditional likelihood, which is maximized only with respect to the β parameters. Once the item parameters estimates have been obtained, they can be used in a next step to obtain estimates of the ability parameters as well. Finally in the

MML approach, the person's effects are considered as independent random variables following a probability density distribution, which is typically chosen to be a normal distribution with mean equal to zero and scale parameter σ^2. In this case, the person's parameters are integrated out of the likelihood function, maximizing it with respect to item parameters and the scale parameter of the normal distribution. Afterwards, predictions of the abilities can be obtained using Empirical Bayes techniques (e.g., Carlin & Louis, 2000).

The MML estimation approach is the most commonly implemented in IRT stand-alone software and it can be implemented (see below) in standard statistical software such as SAS (SAS Institute 2000-2004), R (R Development Core Team, 2009), and STATA (StataCorp., 2009). The estimation methods described above have particular features, advantages and weaknesses that we do not discuss here. For a detailed description of these methods, the reader is referred to Baker and Kim (2004); Embretson and Reise (2000); Tuerlinckx et. al (2004); and Molenaar (1995).

SOFTWARE FOR LIKELIHOOD-BASED IRT ESTIMATION

There are both commercial and free software that have implemented the estimation methods mentioned above. Descriptions of softwares for IRT can be found, for example, in Chapter 13 of Embretson and Reise (2000); and Apendix B of Hambleton, Swaminathan, and Rogers (1991). Many of these softwares are stand alone packages, specifically designed for the estimation of IRT models. However, since various IRT models can be conceived as part of the family of generalized linear mixed models (Rijmen, Tuerlinckx, De Boeck, and Kuppens, 2003), conventional statistical software such as R, STATA, and SAS can be used to estimate them. Since these are less known by users we provide a brief overview of the commands/packages that should be used in these general statistical softwares.

In R, the *lme4* library (Bates, & Sarkar, 2007; Pinheiro & Bates, 2000) uses the *glmer* function to fit generalized linear mixed model which in practice means that one can fit the Rasch model using the MML approach. In addition, there are other R packages which have implemented the estimation of various IRT models. An updated list of these packages can be found at the web address http://cran.r-project.org/web/views/Psychometrics.html and in the special issue "Psychometrics in R" of the freely available Journal of Statistical Software (http://www.jstatsoft.org/v20/).

In STATA, the GLLAMM package (Rabe-Hesketh, Skrondal & Pickles, 2004) can estimate Generalized Linear Latent and Mixed Model, a general family of models to which the Rasch model pertains (see Matschinger, 2006). An annotated example of using GLLAMM to fit the Rasch model can be found at the web site http://www.gllamm.org/aggression.html. In SAS, the NLMIXED procedure (e.g., Sheu, Chen, Su, and Wang, 2005) can be used to implement the MML estimation method. This is the approach taken in De Boeck and Wilson (2004), and followed in the application section of this paper. We compare the results obtained using the Bayesian approach to those obtained using MML estimation as estimated by the NLMIXED procedure of SAS.

Summary of likelihood-based estimation of IRT models

In any statistical model, the probability distribution that is assumed to have generated the data is characterized by a parameter. In the present case, the Bernoulli distribution is governed by the p_{ij} parameter which for IRT models is a function of the parameters θ and β. Likelihood-based methods aim to maximize the likelihood function (i.e., look for the values of θ and β that are the most likely to have characterized the Bernoulli distribution that has generated the data), given the observed data. If the θ's are normally distributed, the statistical model is obtained after integrating out these parameters (i.e., after have averaged with respect to the probability distribution of the person parameter). Then, the marginal likelihood function is maximized with respect to β and the scale parameter σ.

In the next section it will be shown that the Bayesian approach assumes that, before observing the data, all the parameters involved follow probability distributions that are characterized by certain parameters as well. This will permit to incorporate a priori information coming, for instance, from previous studies, about the parameter of interest. If such information is not available, this will no be an impediment to implement the Bayesian approach, as it will always be possible to use *noninformative* priors on the parameters (see later).

THE BAYESIAN METHOD

Suppose that, before observing the data, we know that the parameters of interest follow a probability distribution governed by other parameters. In Bayesian terminology this probability distribution is called the "prior" and the parameters that characterize it, the "hyperparameters". Once the data have been observed, the prior knowledge about the parameter of interest is updated defining what is called the posterior distribution.

The Bayesian method is based on the idea that we can update the prior knowledge about the parameters of interest given the information obtained from the data at hand. Mathematically, the idea is operationalized using the Bayes Theorem or Bayes formula,

$\Pr(A \mid B) = \dfrac{\Pr(B \mid A)\Pr(A)}{\Pr(B)}$. If the probability of the event A is $\Pr(A)$, then the formula states that, after observing an event B, the uncertainty about A can be updated to $\Pr(A \mid B)$, based on the information that B provides, according to the Bayes formula.

The formula also holds for probability density distributions. If $f(\boldsymbol{\alpha})$ represent the prior knowledge about a parameter vector $\boldsymbol{\alpha}$, and the data Y has a density function $f(y \mid \boldsymbol{\alpha})$, then the formula becomes

$$f(\boldsymbol{\alpha} \mid y) = \frac{f(y \mid \boldsymbol{\alpha}) f(\boldsymbol{\alpha})}{f(y)} \propto f(y \mid \boldsymbol{\alpha}) f(\boldsymbol{\alpha}). \qquad (0.4)$$

The symbol $\propto$ means "proportional to" and in this case indicates that the posterior distribution of the parameter $\boldsymbol{\alpha}$ is proportional to the product of the likelihood function and the prior. Note that the vector $\boldsymbol{\alpha}$ contains all the parameters involved in the model. For instance, under the normality assumption for the person's parameters one has that $\boldsymbol{\alpha} = (\theta, \beta, \sigma)$. It should be pointed out that, as we are interested in making inferences about $\boldsymbol{\alpha}$, the term $f(y)$ is a constant in Equation (1.4), and then the inferences can be based on the product $f(y \mid \boldsymbol{\alpha}) f(\boldsymbol{\alpha})$. Note also that, for simplicity of exposition, we have omitted the hyperparameters from (1.4), but it is assumed that the distribution of $\boldsymbol{\alpha}$ is also characterized by a parameter δ and we would write it as $f(\boldsymbol{\alpha} \mid \delta)$.

Bayesian inference is based on the posterior distribution of the parameters of interest. Instead of looking for point values that maximize the likelihood (as it was the case of likelihood-based methods), in the Bayesian approach the inferences are based in the whole posterior distribution of the parameter of interest. In this sense, after drawing samples from the posterior distribution, posterior mean, standard deviations and any other summary can easily be obtained and be more informative than just point estimates.

Drawing samples from the posterior distributions is sometimes straightforward, especially when conjugate priors are used. The prior $f(\boldsymbol{\alpha})$ is said to be conjugate if after multiplying it with the likelihood $f(y \mid \boldsymbol{\alpha})$, the resulting posterior distribution has the same distribution as the prior. In most cases, there are no conjugate priors and one may use numerical algorithms to obtain the samples from the posterior distribution. In order to obtain a sample from the posterior distribution, different iterative methods belonging to the class of Markov Chain Monte Carlo (MCMC) techniques (e.g., Gelman, Carlin, Stern, & Rubin, 2003) have been developed. After the prior distributions are specified for all the parameters in the model, these algorithms generate a Markov Chain which, following an initial burn-in period (often consisting of several thousand iterations), converges to the posterior distribution. Thus, on the condition that convergence is reached, the current state of the chain can be used as a sample from the posterior distribution. Graphical and other diagnostic tools to assess the convergence of the chains, such as the $\hat{R}$ diagnostic (Gelman & Rubin, 1992) are discussed later in the paper.

The technical exposition of the most commonly used MCMC algorithms such as the Gibbs Sampling and the Metropolis-Hastings (Chib and Greenberg, 1995) is out of the scope of this paper. For detailed explanations, the interested reader can consult the books of Gelman et al. (2003) and Robert and Casella (2004). A more applied book showing how to use MCMC methods to complete a Bayesian analysis involving models applied to typical social science data is Lynch (2007).

Fortunately, the software that will be introduced and used later for the Bayesian estimation of IRT models, avoids the explicit programming of the above mentioned algorithms (i.e., Gibbs Sampling or Metropolis-Hastings). It automatically generates the Markov chains after providing the prior distributions and the likelihood function of the model to be estimated.

Bayesian estimation of IRT models

The same two IRT models introduced earlier, the Rasch model and the Linear Logistic Test model (LLTM), will be used to exemplify the Bayesian approach.

Let us denote the prior distributions of β and σ^2 as $f(\beta)$ and $f(\sigma^2)$, respectively. The distribution of θ follows naturally from the model assumptions and it is a normal distribution with mean zero and variance parameter σ^2, denoted as $N(0, \sigma^2)$. Following Equation (1.4), the full posterior distribution is

$$f(\theta, \beta, \sigma \mid y) \propto \prod_{i=1}^{n} \prod_{j=1}^{k} p_{ij}^{y_{ij}} (1 - p_{ij})^{1-y_{ij}} \times N(0, \sigma^2) \times f(\beta) \times f(\sigma^2), \qquad (0.5)$$

where we have assumed independence of priors. Note that this form of the posterior distribution holds for both the Rasch and the LLTM being the only difference in the form of the probability p_{ij} (see Equations (1.1) and (1.2)). The aim is now to obtain samples from the posterior distribution (1.5). All type of summaries such as means, standard deviations, credible intervals and quantiles, among others, can be obtained using these samples. For example, estimates of the β_j parameter will be taken as the mean of its posterior distribution. Thus, after drawing sufficient samples from these posterior distributions, one only has to average these values to obtain an estimate. Note that in the case of the LLTM, under the same distributional assumptions for the person parameters, we are interested in estimating the item properties β_j^* and the scale parameter σ of the normal distribution.

Summary of Bayesian inference for IRT models

Bayesian inference is based on samples from the posterior distribution of the parameters. There are three inputs that play a role in forming the posterior distribution from which we want to draw the samples. First the likelihood function, which is exactly the same as in classical likelihood-based inference, second, the prior distributions for the parameters of interest, third, the hyperparameters governing the prior distributions. In contrast to the likelihood-based approach in which routines converge to a point estimate, using the Bayesian approach one has to monitor the convergence of the Markov chain to the target density, the posterior distribution, before using the samples to conduct inferences.

Software for Bayesian inference

WinBUGS (Spiegelhalter, Thomas, Best, & Lunn, 2003) is a software program for Bayesian analysis of statistical models using MCMC techniques. After the user provides a likelihood and prior distribution, the program automatically draws a sample of all parameters from the posterior distribution. Once the convergence and a good mixing of the chains have been reached, parameter estimates can be obtained and inferences made. The Rasch model and the LLTM were fitted using WinBUGS, which was called from R (R Development Core Team, 2009) by using the R2WinBUGS package (Sturtz, Ligges, & Gelman, 2005). Convergence and the mixing of the chains were assessed using standard graphical techniques (Gilks, Richardson & Spiegelhalter, 1995) and the $\hat{R}$ diagnostic, to be introduced later (Gelman & Rubin, 1992). An alternative to monitor the convergence of the Markov chains is to use the R package CODA (Plummer, Best, Cowles, & Vines, 2006), which implements standard convergence criteria (e.g., Cowles & Carlin, 1996). The use of CODA is explained in Appendix C of Ntzoufras (2009). The main R/WinBUGS code that was used to fit the Rasch and the LLTM is available from the author upon request and described in the Appendix.

A summary of other software for Bayesian analysis can be found in Apendix C of Carlin and Louis (2000). A list of R packages exclusively dedicated to Bayesian analysis can be found in the web page http://cran.r-project.org/web/views/Bayesian.html

APPLICATION

The Bayesian estimation of IRT models using R/WinBUGS will be illustrated using simulated and a real data set. An R function to simulate data according to the Rasch model is provided in the Appendix. The verbal aggression data (De Boeck and Wilson, 2004) is used to illustrate the fit of both the Rasch model and the LLTM model using the Bayesian approach. Bayesian parameter estimates will be compared with those obtained using marginal maximum likelihood estimation.

Recovery study using simulated data

As it was seen before, under the Rasch model, the answers Y_{ij} conditional on the ability follow a Bernoulli distribution with parameter $p_{ij} = \exp(\theta_i - \beta_j)/1 + \exp(\theta_i - \beta_j)$. Thus, we need to specify values for θ and β, and generate samples from a Bernoulli distribution given these values. This is exactly what the R function Rasch.Data() described in Appendix A does. The artificial data set is generated considering $n = 500$ individuals responding to a test composed of $k = 11$ items. The real values of the 11 β parameters are -2.5, -2.0, -1.5, -1.0, -0.5, 0.0, 0.5, 1.0, 1.5, 2.0, and 2.5. For the ability parameters, a normal distribution is assumed with zero mean and variance one. The explanation of how to run the code to obtain the simulated data set can be found in the appendix.

Once the artificial data has been generated, we need to form the posterior distribution in which the Bayesian inference is based. In doing so, prior distributions need to be specified for all parameters in the model. We set as the prior distribution for the difficulty parameter a normal distribution with 0 mean and a large variance of 1000. The huge variance means that this prior represents vague information (is non-informative) about the parameter and then the posterior distribution is almost proportional to the likelihood of the data. Then, when using noninformative priors, it should not be surprising to obtain similar results in comparison with the traditional likelihood based estimation methods. Finally, for the σ parameter, a uniform prior was chosen (Gelman, 2006).

Three Markov chains were run starting from different randomly selected initial values for the parameters of interest in each of the fitted models. This approach helps the researcher monitor convergence and to choose an appropriate burn-in period. Each chain was run with 1000 iterations, and the first half of each chain was discarded as a burn-in stage. Thus, the results that are reported are based on a final sample of 1500 iterations (500 from each of the three chains).

The $\hat{R}$ diagnostic (Gelman & Rubin, 1992) was used to assess convergence. Values of $\hat{R}$ near 1.0 (say, below 1.1) are considered acceptable (Gelman et al., 2003, pp. 296-297). Also, as the inferences will be based in samples from the posterior distribution, one has to be sure of a good mixing of the chains, so graphical tools were used to check the mixing and autocorrelation of the chains. A good mixing of the chains means that the values at each iteration of the algorithm may be drawn from the whole support (i.e., all the possible values the random variable can take) of the distribution. In practice, the generation of the chains and the monitoring tools for convergence is implemented in the software used. The code to implement the above explained setting is explained in the Appendix.

Results of simulation study

First, we use graphical techniques to asses the convergence and mixing of the chains. One commonly used plot shows the history of sampled values at each iteration of the MCMC algorithm. In this plot, the x-axis refers to the iteration number and the y-axis refers to the sampled value. Using this kind of plot, lack of convergence is evidenced if one observes a trend in the sampled values, meaning that the algorithm has not reached a stationary state. Further, if the range of sampled values differs much through different intervals of iterations a bad mixing of the chains is occurring. For example, the first 500 iteration give sampled values between, say, 0.80 and 0.90 whereas iterations 501 to 1000 cover a different range of values. Figures 1 and 2 show the history of the chains' iterations for the first three item parameters and the scale parameter σ, respectively.

Figure 1. *History of chain iterations for the first three item parameters*

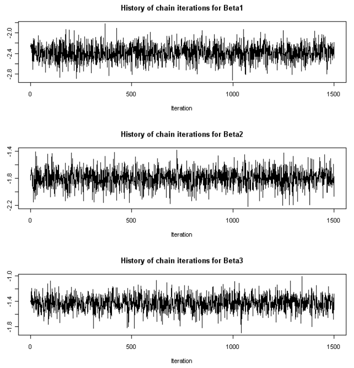
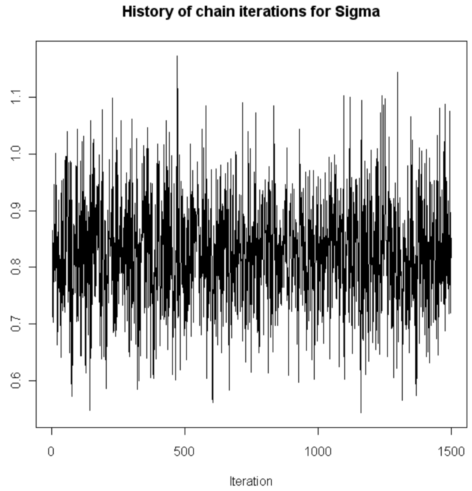


Figure 2. *History of iterations for the σ parameter*



From the figures it can be seen that the chains don't follow a clear trend meaning that they would have converged at a stationary stage. Moreover, the range of sampled values throughout the iterations is homogeneous meaning that the chains are mixing very well.

We also used the $\hat{R}$ diagnostic to assess the convergence of the chains. The values of $\hat{R}$ for all the parameters was less than or equal to 1.03 (see R output) for all fitted models so we concluded that convergence had been established.

Once we are sure that the chains have converged, we can use samples from the posterior distribution to obtain summaries. Table 1 shows the recovery results of parameter estimates. The values in column WinBUGS correspond to the posterior mean of the parameter (i.e, the mean of the 1500 values drawn from the posterior distribution).

Table 1: *Real value and parameter estimates: MML and WinBUGS results*[(b)]

Item	Real β value	MML[(a)]	WinBUGS
1	-2.5	-2.43 (0.16)	-2.39 (0.16)
2	-2.0	-1.83 (0.13)	-1.80 (0.13)
3	-1.5	-1.46 (0.12)	-1.43 (0.12)
4	-1.0	-0.88 (0.11)	-0.86 (0.11)
5	-0.5	-0.44 (0.11)	-0.43 (0.10)
6	0.0	0.04 (0.11)	-0.03 (0.11)
7	0.5	0.42 (0.11)	0.41 (0.10)
8	1.0	0.91 (0.11)	0.89 (0.11)
9	1.5	1.44 (0.12)	1.42 (0.12)
10	2.0	1.91 (0.14)	1.88 (0.13)
11	2.5	2.34 (0.15)	2.32 (0.15)

[(a)] MML estimates were obtained using the rasch() function from the R package ltm (Rizopoulos, 2006).

[(b)] Standard errors and posterior standard deviations in parenthesis.

From the table it can be seen that the recovery is very good. The parameter estimates obtained by WinBUGS were very close to the real values. Moreover, the WinBUGS results are very similar to those obtained with MML estimation, the classical likelihood-based approach. This shows that when using the Bayesian method estimation one can obtain the same level of accuracy as traditional likelihood-based methods, provided that non-informative priors have been used.

Real data application: Rasch model

We use the "verbal aggression" data to exemplify the Bayesian estimation of both the Rasch and the LLTM model. A full description of the data can be found in De Boeck and Wilson (2004).

The same number of chains as used in the simulation study was run and convergence was assessed also in same way. The WinBUGS code to fit the Rasch model is shown in the Appendix.

History plots of items and the scale parameter (not shown) gave evidence of convergence and a good mixing of the chains. Also the $\hat{R}$ values were all less than 1.004, meaning that approximate convergence has been reached.

The item parameter estimates and the scale parameter of the random effects distribution are shown in Table 2. This table contains the results using various softwares, and appears in Chapter 11 of De Boeck and Wilson (2004). We have added the column "WinBUGS" in order to compare the results with the other software.

Table 2: *Estimates of the item parameters and their standard error, and the scale parameter of the random effects distribution and its standard error (taken from De Boeck and Wilson, 2004: the WinBUGS column has been added). Reproduced with permission of Springer Science+Business Media, LLC"*

Item	MLwiN	HLM	GLIMMIX	NLMIXED	GLLAMM	MIXOR	WinBUGS
1	-1.22 (0.16)	-1.17 (0.22)	-1.17 (0.16)	-1.22 (0.16)	-1.23 (0.16)	-1.23 (0.19)	-1.21 (0.16)
2	-0.56 (0.15)	-0.54 (0.21)	-0.54 (0.15)	-0.57 (0.15)	-0.57 (0.15)	-0.57 (0.18)	-0.55 (0.16)
3	-0.08 (0.15)	-0.08 (0.21)	-0.08 (0.15)	-0.09 (0.15)	-0.09 (0.15)	-0.08 (0.17)	-0.07 (0.15)
4	-1.74 (0.17)	-1.67 (0.22)	-1.67 (0.17)	-1.75 (0.17)	-1.76 (0.17)	-1.75 (0.20)	-1.74 (0.18)
5	-0.71 (0.15)	-0.68 (0.21)	-0.68 (0.15)	-0.71 (0.15)	-0.71 (0.15)	-0.71 (0.15)	-0.70 (0.15)
6	-0.01 (0.15)	-0.01 (0.21)	-0.01 (0.15)	-0.02 (0.15)	-0.02 (0.15)	-0.02 (0.18)	-0.01 (0.15)
7	-0.53 (0.15)	-0.51 (0.21)	-0.51 (0.15)	-0.53 (0.15)	-0.54 (0.15)	-0.53 (0.17)	-0.52 (0.15)
8	0.69 (0.16)	0.66 (0.21)	0.66 (0.15)	0.68 (0.15)	0.68 (0.15)	0.68 (0.15)	0.71 (0.16)
9	1.53 (0.17)	1.46 (0.22)	1.46 (0.17)	1.52 (0.17)	1.52 (0.17)	1.52 (0.19)	1.55 (0.18)
10	-1.08 (0.16)	-1.03 (0.21)	-1.03 (0.16)	-1.09 (0.16)	-1.09 (0.16)	-1.09 (0.18)	-1.07 (0.16)
11	0.35 (0.15)	0.33 (0.21)	0.33 (0.15)	0.34 (0.15)	0.34 (0.15)	0.34 (0.17)	0.36 (0.15)
12	1.04 (0.16)	1.00 (0.21)	1.00 (0.16)	1.04 (0.16)	1.04 (0.16)	1.04 (0.19)	1.06 (0.17)
13	-1.22 (0.16)	-1.17 (0.22)	-1.17 (0.16)	-1.23 (0.16)	-1.23 (0.16)	-1.23 (0.19)	-1.21 (0.16)
14	-0.39 (0.15)	-0.37 (0.21)	-0.37 (0.15)	-0.40 (0.15)	-0.39 (0.15)	-0.40 (0.19)	-0.38 (0.16)
15	0.87 (0.16)	0.83 (0.21)	0.83 (0.15)	0.87 (0.16)	0.87 (0.16)	0.87 (0.18)	-0.88 (0.15)
16	-0.87 (0.16)	-0.83 (0.21)	-0.83 (0.15)	-0.87 (0.15)	-0.88 (0.16)	-0.88 (0.18)	-0.87 (0.16)
17	0.06 (0.15)	0.05 (0.21)	0.05 (0.15)	0.05 (0.15)	0.06 (0.15)	0.05 (0.18)	0.06 (0.15)
18	1.48 (0.17)	1.42 (0.22)	1.42 (0.16)	1.48 (0.17)	1.48 (0.17)	1.48 (0.17)	1.50 (0.17)
19	0.21 (0.15)	0.20 (0.21)	0.20 (0.15)	0.21 (0.15)	0.21 (0.15)	0.21 (0.17)	0.22 (0.15)
20	1.50 (0.17)	1.44 (0.22)	1.44 (0.17)	1.50 (0.17)	1.50 (0.17)	1.50 (0.19)	1.52 (0.16)
21	2.96 (0.23)	2.84 (0.27)	2.84 (0.22)	2.97 (0.23)	2.97 (0.23)	2.98 (0.25)	3.00 (0.23)
22	-0.71 (0.15)	-0.68 (0.21)	-0.68 (0.15)	-0.71 (0.15)	-0.71 (0.15)	-0.71 (0.18)	-0.70 (0.16)
23	0.38 (0.15)	0.37 (0.19)	0.37 (0.15)	0.38 (0.15)	0.38 (0.15)	0.38 (0.18)	0.39 (0.16)
24	1.99 (0.18)	1.91 (-)	1.91 (0.18)	2.00 (0.18)	2.00 (0.18)	2.00 (0.20)	2.02 (0.18)
Variance Of the intercept	1.87 (0.17)	1.69 (0.15)	1.70 (0.17)	1.98 (0.21)	1.98 (0.21)	1.98 (-)	1.95 (0.20)

It can be seen that the results are very similar when comparing the Bayesian approach and the more classical approaches. As it was mentioned before, this similarity in the results is in a way not surprising, since using noniformative priors, the posterior distribution is almost proportional to the likelihood function, in which the likelihood-based method is fully based.

Second example: The LLTM

Chapter 2 of De Boeck and Wilson (2004) shows the fit of the LLTM model for the verbal aggression data. We fit the model using the WinBUGS program. After checking convergence and good mixing of the chains, the parameter estimates obtained are show in Table 3. This table show the estimates using both the NLMIXED procedure from SAS, as reported by De Boeck and Wilson (2004) and WinBUGS.

Tabla 3: *Item properties parameter estimates for the LLTM using NLMIXED (MML) and WinBUGS*

Item property	MML$^{(a)}$	WinBUGS
1		1.33 (0.10)
2	-0.99 (0.07)	-0.99 (0.07)
3	-2.04 (0.07)	-2.05 (0.07)
4	0.67 (0.06)	0.67 (0.06)
5	-1.03 (0.06)	-1.03 (0.06)

De Boeck and Wilson (2004) used a mixed coding scheme, so that the value of item property 1 is not comparable to the WinBUGS results.

Again, the values are in good correspondence (almost identical). The estimate of the σ^2 parameter is reported to be 1.86 (0.20) (De Boeck and Wilson, 2004) and the corresponding estimated value in WinBUGS is 1.85 (0.19). It can be seen that the Bayesian estimates are practically identical to the ones obtained using MML estimation.

CONCLUSIONS

This paper intended to serve as a brief introduction to the Bayesian method in the field of IRT models and, mainly, to be a tutorial for the implementation of estimation algorithms using freely available software. The Bayesian approach for the estimation of two IRT models (the Rasch and the LLTM models) was presented. This estimation approach allows drawing samples from the posterior distributions of the parameters we are interested on; it provides not only the mean of the posterior but the complete distribution, permitting to obtain different kinds of summary statistics such as quantiles, means, etc.

Besides the advantage of incorporating prior knowledge of the parameter, the Bayesian method is a computationally more convenient way to estimate IRT models. The CML, MML and JML methods have strengths and weaknesses but as models are being extended to accommodate more random effects (i.e., multidimensional traits or random item effects), they become computationally infeasible.

The introduction and brief summary of the main ideas behind the Bayesian approach given in this paper is not at all exhaustive, and this is why many good references were cited to guide the reader in some topics.

Although only parametric models were considered here, there are methodologies and software for the implementation of Bayesian semi-parametric models as well. The interested reader is referred to Miyazaki and Hoshino (2009), and Jara (2007).

We have not considered here the issue of model selection, model checking, and goodness of fit, thought there are Bayesian counterparts that account for it. For model selection under the Bayesian approach, one may calculate Bayes factors (Kass & Raftery, 1995) and select the model with the largest posterior probability given the data. Other possibility is the use of the deviance information criteria (DIC; Spiegelhalter, Best, Carlin & Van der Linde, 2002), which is a Bayesian counterpart of the more classical measures AIC and BIC. In assessing goodness of fit, one may use posterior predictive checks (PPC; Gelman et. al, 2003) using samples from the posterior distribution.

The Rasch and the LLTM models are only two of many other IRT models that can be fitted using the Bayesian approach. Examples of more complex models in the family of IRT models that have been estimated using Bayesian methods can be found in Janssen et al. (2000, 2004); González, De Boeck, & Tuerlinckx (2008); González, Tuerlinckx and De Boeck (2009); among others. These works give evidence that the Bayesian inference approach shows to be promising for studying complex models in psychological research.

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Appendix

R code to generate artificial data according to the Rasch model

```
Rasch.data<-function(n,k,beta){                                        #1#
set.seed(1)                                                            #2#
theta<-rnorm(n,0,1)                                                    #3#
probs<-matrix(0,nrow=n,ncol=k)                                         #4#
dat<-matrix(0,nrow=n,ncol=k)                                           #5#
for(i in 1:n){                                                         #6#
 for(j in 1:k){                                                        #7#
  probs[i,j]<-exp(theta[i]-beta[j])/(1+exp(theta[i]-beta[j]))          #8#
  dat[i,j]<-rbinom(1,1,probs[i,j])                                     #9#
}                                                                      #10#
}                                                                      #11#
return(dat)                                                            #12#
}                                                                      #13#
```

In #1# we assign the name Rasch.data specifying that it is a function of the number of persons n, the number of items k, and the real values of the β parameters. In #3# it is indicated that the values of the abilities are drawn from a normal distribution of zero mean and standard deviation 1. In #4# and #5# matrices are created to store the probabilities p_{ij} and the data matrix of zeros and ones, whose elements are the observations y_{ij}.

Once the function has been compiled in R, a call to generate data is as follows

```
data.set1=Rasch.data(500,11,seq(-2.5,2.5,0.5))
```

where data.set1 is the object in which the data are stored.

R-WinBUGS code to fit the models

We present the used code to fit the models presented in the paper. This code can easily be adapted to fit other models. For a detailed explanation of the R2WinBUGS package the reader is referred to Sturtz, Ligges, and Gelman (2005).

Preliminaries

1. Install WinBUGS if it is not already installed in your system. The program and all the information for installation can be found at http://www.mrc-bsu.cam.ac.uk/bugs/
2. Install R and the R2WinBUGS package if they are not already installed. R and all available packages can be found at http://www.r-project.org/
3. Write the WinBUGS code of the model in the file file.txt (see below)
4. Run the R script to call WinBUGS from R (see below)

WinBUGS code

Copy and paste the following WinBUGS code to create the Rasch Model rd.txt file.

```
model;                                                                 #1#
{                                                                      #2#
for(i in 1:n){                                                         #3#
theta[i]~dnorm(0,tau)                                                  #4#
 for(j in 1:k){                                                        #5#
  p[i,j]<-exp(theta[i]-beta[j])/(1+exp(theta[i]-beta[j]))              #6#
  x[i,j]~dbern(p[i,j])                                                 #7#
}                                                                      #8#
}                                                                      #9#
for(j in 1:k){                                                         #10#
beta[j]~dnorm(0,0.001)                                                 #11#
}                                                                      #12#
tau<-pow(sigma,-2)                                                     #13#
sigma~dunif(0,100)                                                          #14#
sigma.theta<-1/tau                                                          #15#
}                                                                           #16#
```

From lines 3 to 9 the likelihood of the model is specified. From lines 10 to 12, the prior distribution for the β parameters is specified. From lines #13# to #15# the prior for the σ parameter is specified

Fitting the Rasch model using *ltm* to obtain MML estimates to be compared with WinBUGS results

```
library(ltm)
rasch.fit=rasch(data.set1, constraint = cbind(ncol(data.set1) + 1, 1))
summary(rasch.fit)[1]
```

R code to call WinBUGS from R via the R2WinBUGS package

```
library(R2WinBUGS)                                                                    #1#
dat=read.table("data.set1.txt",header=F)                                              #2#
n=nrow(dat)                                                                           #3#
k=ncol(dat)                                                                           #4#
x=as.matrix(dat)                                                                      #5#
data=list("n","k","x")                                                                #6#
inits=function(){                                                                     #7#
list(theta=rnorm(n,0,1),beta=rnorm(11,0,1),sigma=runif(1))                            #8#
}                                                                                     #9#
parameters=c("beta","sigma.theta")                                                   #10#
mod1<-bugs (data, inits, parameters, "Rasch Model rd.txt",n.chains=3, n.iter=1000,   #11#
n.thin=1)                                                                            #12#
```

In #1# the R2WinBUGS library which allows to call WinBUGS from R is loaded. In #2# the data set in file “data.set1.txt” is loaded. In #3# and #4# the number of rows and columns are specified. Note that these values coincide with the number of individuals and the number of items, respectively. From #7# to #9#, initial values for the algorithm are generated. In #10# it is specified which are the parameter we want to obtain samples from its posterior distributions. In #10# and #11# the bugs function makes call to WinBUGS, using the arguments previously described. The n.chains argument indicate that 3 parallel chains are being run, each with n.iter=1000 iterations. By default, half of each chain is discarded as a burn-in